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NOTE

AN ALGEBRAIC THEORY OF GRAPH FACTORIZATION

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An algebraic theory of graph factorization is introduced. For a factor h, a graph $G(h)$ is constructed whose structure contains information about h-factorability. The 1-factorable and cycle factorable graphs over $\mathbb{Z}_2$ are characterized, and properties of the corresponding graph $G(h)$ are obtained.

1. Algebraic factorization

A graph G is usually defined to be the sum of factors $\{G_i\}$ if it is their line disjoint union. An algebraic version of graph factorization can be formulated as follows. Let K_p be the complete graph on p points and $E(K_p)$ the set of directed lines of K_p. Let R be a ring and $S(K_p)$ the module of 1-chains $\sum a_i x_i$ where $a_i \in R$, $x_i \in E(K_p)$ and $(v, v') = -(v', v)$ for all $(v, v') \in E(K_p)$. For 1-chains we say that g is the *sum of factors* $\{g_i\}$ if $g = \sum g_i$. A permutation of the points of K_p naturally induces a permutation of the elements of $S(K_p)$. Two 1-chains are considered *isomorphic* if there is such a permutation taking one to the other. Any 1-chain isomorphic to a given 1-chain h is called an *h-factor*. Any element of a submodule $M \subseteq S(K_p)$ is said to be M-factorable. In particular, an element in the submodule generated by the set of h-factors is said to be *h-factorable*. Thus a 1-chain g is h-factorable if and only if g is the sum of h-factors.

For a submodule $M \subseteq S(K_p)$, two 1-chains are M-equivalent if they differ by an M-factor. More precisely, g and g' are called M-equivalent if there exists 1-chains $\bar{g}$ and $\bar{g}'$, isomorphic to g and g' respectively, such that $\bar{g} - \bar{g}'$ is M-factorable. For a 1-chain h, h-equivalence is defined similarly. The submodule M of h-factorable 1-chains is one of the h-equivalence classes. In general, each h-equivalence class is the union of elements of $S(K_p)/M$. Denote the set of h-equivalence classes by $S(p, h)$. Define the *structure graph* $G(p, h)$ as follows. The points of $G(p, h)$ are the 1-chains of $S(K_p)$ up to isomorphism, and two points are adjacent if and only if they differ by a single h-factor. Properties of h-factorization are reflected in the structure of $G(p, h)$. The set of connected components of $G(p, h)$ corresponds to $S(p, h)$. If the diameter of the component of $G(p, h)$ containing the null graph is d, then any h-factorable 1-chain is the sum of at most d h-factors.

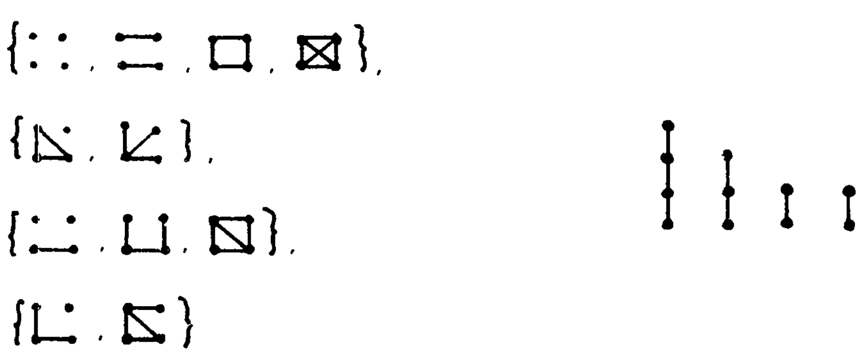
When R is the ring $\mathbb{Z}_2$ of integers modulo 2, a 1-chain $\sum a_i x_i$ can be identified with a subgraph of K_p, namely the subgraph consisting of the lines x_i for which $a_i \neq 0$. In this case we may speak of subgraphs as being h-factors or h-factorable over $\mathbb{Z}_2$. In general we say that a subgraph G of K_p is *h-factorable over R* if there is an h-factorable 1-chain $\sum a_i x_i$ with $a_i \neq 0$ if and only if x_i is a line of G.

Example 1. Let $R = \mathbb{Z}_2$ and h a single line. Then each line of K_p is an h-factor and all subgraphs of K_p are h-factorable. Hence $G(p, h)$ has only one component. Two points of $G(p, h)$ are adjacent if the corresponding subgraphs differ by exactly one line. At the other extreme consider $h = K_p$. Only the null graph and K_p are K_p-factorable over $\mathbb{Z}_2$. Each connected component of $G(p, K_p)$ is a K_2 corresponding to a subgraph of K_p and its complement.

The factors most often discussed in the classical setting are n-factors. As usual call a regular spanning subgraph f_n, of degree n, an n-factor. The terms f_n-factorable and n-factorable will be used interchangably.

Example 2. When $p = 4$, the four 1-factorable graphs over Z_2 are shown in Fig. 1. The three additional equivalence classes in $S(4, f_1)$ and the structure graph $G(4, f_1)$ are also shown.

Example 3. Consider the directed cycle c_3 of length 3. It can be shown that a cubic graph G is c_3-factorable over $\mathbb{Z}_3$ if and only if G is bipartite. A cubic graph is c_3-factorable over $\mathbb{Z}_4$ if and only if G has a proper three coloring of the lines. It is easy to show that a graph with a bridge is not c_3-factorable over $\mathbb{Z}_n$ for any $n \geq 2$. The Petersen graph is the smallest bridgeless cubic graph that is not c_3-factorable over $\mathbb{Z}_4$. To characterize c_3-factorable graphs over $\mathbb{Z}_3$ and $\mathbb{Z}_4$ appears difficult [2]. It is also unknown whether every bridgeless graph has a c_3-factorizaton over $\mathbb{Z}_5$.



$S(4, f_1)$ $G(4, f_1)$

Fig. 1. Structure of 1-factorability in K_4 over $\mathbb{Z}_2$.

2. Factorization over $\mathbb{Z}_2$

In this section factorization is considered over the ring $\mathbb{Z}_2$. We state, without proof, only results concerning 1-factorable and cycle factorable graphs. For a fixed $2 < m \leq p$, c_m denotes a cycle of length m and p_m a path of length m. The null graph is written $\emptyset$ and rg is the union of r copies of the graph g. The symbol $\cup$ denotes disjoint union. Also f denotes a 1-factor.

Theorem 1. *A subgraph g of K_{2n} with q lines is 1-factorable if and only if either* (1) $\deg(v)$ *is even for all points v of g and q is even or* (2) $\deg(v)$ *is odd for all v and $q+n$ is even.*

Theorem 2. *The graph $G(2n, f)$ has $2[\frac{1}{2}n]+2$ connected components, and representatives of the classes of $S(2n, f)$ are $\emptyset$, K_3, $p_1, 2p_1, \ldots, [\frac{1}{2}n]p_1$, p_2, $p_2 \cup p_1$, $p_2 \cup 2p_1, \ldots, p_2 \cup ([\frac{1}{2}n]-1)p_1$.*

Analogous theorems hold for factorability with respect to cycles.

Theorem 3. *A subgraph g of K_p with q lines is c_m-factorable over $\mathbb{Z}_2$ if and only if either* (1) *m is odd and* $\deg(v)$ *is even for all v or* (2) *m and q are even and* $\deg(v)$ *is even for all v.*

Theorem 4. *The graph $G(p, c_m)$ has $[\frac{1}{2}p]+1$ connected components if m is odd and $2[\frac{1}{2}p]+2$ connected components if m is even. For m odd, representatives of the classes of $S(p, c_m)$ are $\emptyset, K_3, p_1, 2p_1, \ldots, [\frac{1}{2}p]p_1$; for m even, representatives of $S(p, c_m)$ are $\emptyset, K_3, p_1, 2p_1, \ldots, [\frac{1}{2}p]p_1, p_2, p_2 \cup p_1, \ldots, p_2 \cup [\frac{1}{2}(p-3)]p_1$ and also $K_{1,3} \cup (\frac{1}{2}p-2)p_1$ when p is even.*

In [1] K_{2n} is expressed as a sum of $(2n-1)$ 1-factors. It is likely that this result can be generalized.

Conjecture 1. Any 1-factorable subgraph of K_{2n} over $\mathbb{Z}_2$ is the sum of at most $(2n-1)$ 1-factors.

This conjecture follows from the stronger statement:

Conjecture 2. Let f be a 1-factor of K_{2n}. Over $\mathbb{Z}_2$ the diameter of each connected component of $G(2n, f)$ is at most $2n-1$.

References

[1] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1972).

[2] R. Isaacs, Infinite families of non-trivial trivalent graphs which are not Tait-colourable, Amer. Math. Monthly (82) 3 (1975) 221–239.